Vibration Analysis of the Plates Subject to Distributed Dynamic Loads by Using Spectral Element Method

Usik Lee* **and Joonkeun Lee****

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A spectral element method (SEM) is introduced for the vibration analysis of rectangular plates under distributed dynamic loads. In this paper, the spectral plate element matrix (often called the dynamic stiffness matrix) is formulated from the relations between the forces and displacements along the opposite two parallel edges. The distributed dynamic load is discretized into a sequence of equivalent line loads. The plate is then considered as a connection of two spectral plate elements with the joint node line along which the equivalent line load acts. The spatial coordinate dependence of each equivalent line load is then removed through the spatial Fourier transformation so that the plate (2-D) problem becomes a simplified equivalent beam-like (1-D) problem. The remaining solution procedure is therefore the same as that used for beam problems. Numerical tests show that the present SEM provides very accurate solutions when compared to finite element solutions.

Key Words: Spectral Element Method, Vibration Analysis, Rectangular Plates, Distributed Dynamic Loads

1. Introduction

Structural dynamics has become an important research subject in engineering during the last for decades, and many structural analysis methods have accordingly been developed. The Finite element method (FEM) is certainly one of the most commonly used methods. However, it is well-known that a sufficiently large number of finite elements is recessary in order to obtain reliable dynamic response of a structure, especially at high frequencies. Furthermore, the modal analysis, usually used in conjunction with FEM, is limited to the frequency regimes where relative spacing of natural frequencies remains large compared to the relative parameter uncertainty (Plaut and Huseyin 1973 ; von Flotow 1987). Thus, alternatives to modal analysis of linear structural dynamics, applicable to the high frequency regime, have been considered by many researchers.

Structural vibration can be considered as the superposition of incoming and outgoing traveling (elastic) waves (von Flotow 1986). Hence accurate dynamic response of a structure may be obtained from the exact wave modes (or shape functions) of the structural dynamic equations in contrast to the finite element solutions based on approximated shape functions.

In the literature, there have been efforts to obtain the dynamic response of discrete systems by efficiently utilizing the FFT and inverse FFT (or IFFT) algorithms (e.g., Mehl and Miles 1995). This type of solution approach is known as the *spectral analysis method* (SAM). Doyle (1986) is one of the first to apply SAM to wave propagation analysis of one-dimensional structures. He assumed the spectral form of dynamic response and applied the FFT to transform the external dynamic loads into frequency domain representations. He obtained the dynamic responses in the time domain by summing up the spectral components of solutions (or wave modes) and then by inverse Fourier transforming

* Department of Mechanical Engineering, Inha University
** Mechatronics R & D Center, LG Cable & Machinery

the result into the time domain via the IFFT. Application of FFT and IFFT algorithms in the solution procedure makes it possible to efficiently take into account as many high frequency wave modes as needed, which may improve the solution accuracy significantly at high frequency ; this may not be true in modal analysis based on FEM.

To extend SAM application to multiply connected structures, Doyle (1987) introduced the concept of the spectrally formulated finite element (simply, spectral element). The Spectral element matrix of a structure, which is often called the dynamic stiffness matrix (e.g., Leung 1993 ; Banerjee 1997), is formulated directly from the exact wave solutions or shape functions of a structure by treating the mass distribution exactly. Thus, in contrast to the conventional finite element in which mass distribution is approximately modeled, the spectral element treats the dynamic characteristics within a structure exactly as long as the mathematical model of the structural element is valid in a specified frequency regime. We may also benefit from the spectral element formulation in that the spectral elements can be readily assembled in a completely analogous way to that used in conventional FEM.

Combining the aforementioned promising characteristics of SAM with the structural discretization and assembling features of FEM leads to a new innovative solution method known as the *spectral element method* (SEM) (Doyle, 1989). Applications of SEM to the structural dynamics problems can be found in recent papers for the multiply connected Timoshenko beam (Gopalakrishhan *et al.* 1992), for plane truss structures (Horr and Schmidt 1995 ; Lee and Kang, 1995 ; Lee and Lee 1996), and for beams and plates subjected to dynamic concentrated loads (Lee et. al. 1996 ; Lee and Lee, 1997)

However, most SEM applications have been confined to one-dimensional simple structures subject to concentrated dynamic loads. This is because special techniques are required to cope with the distributed dynamic loads and the multi-directional wave characteristics in multi-dimensional structures. Lee and Lee (1996) is likely to be the first to modify the conventional SEM and extend it to structures with distributed dynamic loads. However, this modified SEM is applicable only to one-dimensional structures, i. e., beams. In the literature, Danial and Doyle (1995) recently considered a semi-infinite plate problem based on the SAM approach, but a SEM technique applicable to finite plates was not developed. This is probably because the dispersive relation between two distinct wave numbers of a finite plate is not as simple as the case for beams. Special techniques may thus be required to develop SEM for finite plates.

The objectives of this paper are : (1) to develop a SEM technique for finite rectangular plates subject to distributed dynamic loads ; and (2) to verify the accuracy of the present SEM through some numerical examples.

2. Formulation of Spectral Plate Element Matrix

Unlike beams, two-dimensional structures have very complicated dispersive characteristics. Thus, a special SEM technique is required for two-dimensional structures. Since the SEM technique developed in this paper can be extended to the rectangular plates with arbitrary boundary conditions with some modification, the present discussion will be confined to rectangular plates with simply-supported boundary conditions at two opposite parallel edges, which are known to have analytical or closed-form solutions. Consider a thin rectangular plate subject to a distributed dynamic load $f(x,y,t)$. Assuming a small amplitude vibration $w(x,y,t)$, the dynamic equations of motion are given as (Szilards, 1974) :

$$D\nabla^4 w + m\frac{\partial^2 w}{\partial t^2} = f(x,y,t) \tag{1}$$

where D and m are the flexural rigidity and the mass density per unit area of the plate, respectively.

Let the displacement history at an arbitrary point in the plate have the spectral representation (Doyle, 1989)

$$w(x, y, t) = \sum_n \overline{w}_n(x, y ; \omega_n) e^{i\omega_n t}$$

$$=\sum_n \bar{X}_n(x;\omega_n)\,\bar{W}_n(y;\omega_n)\,e^{i\omega_n t} \quad (2)$$

In the remainder of our development the subscripts n not be written out explicitly for brevity. The spectral shape functions $\bar{X}(x)$ and $\bar{W}(y)$ in the x-and y-directions must also satisfy relevant boundary conditions. The dispersive relation for a plate is given as

$$(k_x^2+k_y^2)^2=\Omega^4 \quad (\Omega^2=\omega\sqrt{m/D}) \qquad (3)$$

where k_x and k_y are the wave numbers in the x-and y-directions, respectively. From Eq. (3), k_y can be expressed in terms of k_x and ω, and k_x in terms of k_y and ω. Thus, in this paper, the wave characteristics within a plate will be expresented in terms of k_x and ω.

For rectangular plates which have simply-supported boundary conditions at two opposite edges of $x=0$ and $x=L$, the spectral shape function $\bar{X}(x)$ can be readily derived as

$$\bar{X}(x;k_x)=e^{ik_xx}-e^{-ik_xx} \qquad (4)$$

with the characteristic wave number k_x defined as

$$k_x=\frac{n\pi}{L} \quad (n=1,2,3,\cdots) \qquad (5)$$

In this case, the spectral shape function $\bar{W}(y)$ is found to be

$$\bar{W}(y;k_x,\omega)=B_1e^{-ik_yy}+B_2e^{-k_yy}+B_3e^{ik_yy}+B_4e^{k_yy} \qquad (6)$$

where

$$k_y=\sqrt{\Omega^2-k_x^2} \qquad (7)$$

The spectral displacement $\bar{w}(x,y)$ can be rewritten from Eqs. (2), (4) and (6), omitting the subscripts for brevity, as follows :

$$\bar{w}(x,y;k_x,\omega)=\bar{X}(x;k_x)\,\bar{W}(y;k_x,\omega)=\bar{X}(x;k_x)\,[Q(y;k_x,\omega)]\{B_1\ B_2\ B_3\ B_4\}^T \quad (8)$$

where

$$[Q(y;k_x,\omega)]=\left[e^{i\sqrt{\Omega^2-k_x^2}\,y}\ e^{-i\sqrt{\Omega^2-k_x^2}\,y}\ e^{\sqrt{\Omega^2-k_x^2}\,y}\ e^{-\sqrt{\Omega^2-k_x^2}\,y}\right] \qquad (9)$$

To derive the spectral element matrix for a plate, the same procedure that is used for beams in the previous works by the authors will be used. Figure 1 shows a spectral plate element which has dimensions L and l in the x-and y-directions,

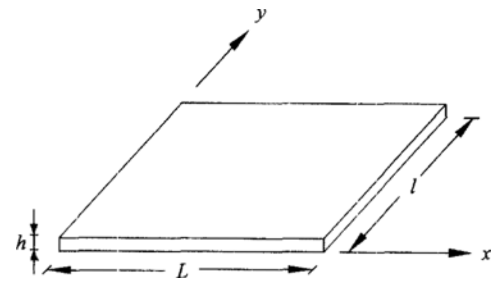


Fig. 1 Geometry of a rectangular plate element.

respectively. The spectral nodal DOFs on the boundary edges at $y=0$ and $y=l$ can be expressed from Eq. (8) in the form of

$$\{\bar{w}_0\bar{w}_0'\bar{w}_l\bar{w}_l'\}^T=X(x)\,[\alpha(k_x,\omega)]\{B_1\ B_2\ B_3\ B_4\}^T \qquad (10)$$

where (′) indicates the partial derivative with respect to y. In Eq. (10), $\bar{w}_0=\bar{w}(0)$, $\bar{w}_0'=\bar{w}'(0)$, $\bar{w}_l=\bar{w}(l)$, and $\bar{w}_l'=\bar{w}'(l)$ are defined, and the (4×4) matrix $[\alpha(k_x,\omega)]$ is tabulated in the Appendix. The spectral shear force and bending moments on the boundary edges at $y=0$ and $y=l$ can be written as

$$\bar{V}=D\left[\frac{\partial^3\bar{w}}{\partial y^3}+(2-\nu)\frac{\partial^3\bar{w}}{\partial x^2\partial y}\right] \qquad (11)$$

$$\bar{M}=D\left[\frac{\partial^2\bar{w}}{\partial y^2}+\nu\frac{\partial^2\bar{w}}{\partial x^2}\right] \qquad (12)$$

where ν is Poisson's ratio.

Using Eqs. (11) and (12), the spectral nodal moments and forces can be expressed in the form

$$\{\bar{V}_0\ \bar{M}_0\ \bar{V}_l\ \bar{M}_l\}^T=X(x)\,[\beta(k_x,\omega)]\{B_1\ B_2\ B_3\ B_4\}^T \qquad (13)$$

where the (4×4) matrix $[\beta(k_x,\omega)]$ is also tabulated in the Appendix. Combining Eq. (10) and (13) results in

$$\{\bar{V}_0\ \bar{M}_0\ \bar{V}_l\ \bar{M}_l\}^T=[\sigma(k_x,\omega)]\{\bar{w}_0\ \bar{w}_0'\ \bar{w}_l\ \bar{w}_l'\}^T \qquad (14)$$

where $[\sigma]=[\beta]\ [\alpha]^{-1}$ is the spectral element matrix for plates. The spectral plate elements can be assembled in an analogous way to the method used in FEM.

3. Eouivalent Beam-Like Problem Representation

Distributed dynamic loads acting on a rectangular plate can be discretized into many equivalent line loads. Figure 2(a) shows an example of a square plate subjected to a centered uniform square load $f(x,y,t)$. In this example, the distributed load is discretized into a series of equivalent line loads which are parallel to the x-axis. Accordingly the plate should also be discretized into two spectral plate elements with the node line parallel to x-axis, as shown in Fig. 2(b). The Dynamic response of the plate can be obtained by summing up all the dynamic responses due to each of the equivalent line loads.

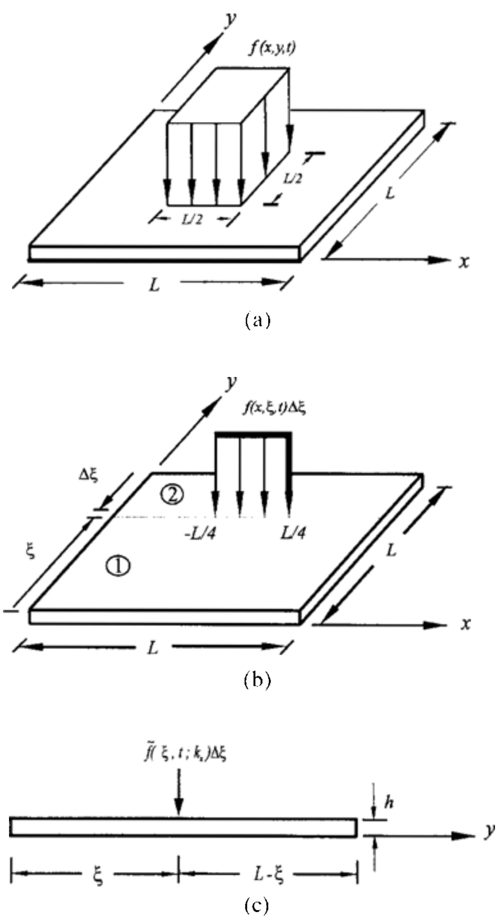


Fig. 2 Spectral element discretization procedure for plate problems : (a) a square plate subject to a centered square load, (b) discretization into two spectral plate elements with a line load, and (c) its equivalent one-dimensional problem representation.

The equivalent line load at $y=\xi$ has magnitude $f(x, \xi, t)\Delta\xi$ and is a function of spatial coordinates x and time t. The equivalent line load can be Fourier transformd with respect to x as well as with respect to time t to yield $\bar{f}(\xi ; k_x, \omega)\Delta\xi$, where the symbol bar($\bar{}$) represents the result of Fourier transform with respect to x. This procedure may remove the x-axis dependence of the equivalent line load and the plate (2-D) problem is transformed into an equivalent beam (1-D) problem, as illustrated in Fig. 2(c). We benefit from this procedure in that the general solution procedure developed for beams in the previous works can be utilized to solve the vibration problems of plates via the equivalent beam problem representation.

4. General Solution Procedures

The foregoing structural and load discretization procedures represent a plate as a connection of two spectral plate elements with an equivalent line load acting along the connection node line, as shown in Fig. 2(b). Equation (14) shows that the spectral element matrix is formulated only in terms of the wave number k_x at a specified frequency ω. Thus, by using the spatial Fourier transformed equivalent line load $\bar{f}(\xi ; k_x, \omega)\Delta\xi$, two spectral plate elements can be assembled to yield a global spectral matrix equation for each equivalent line load. Applying the simply supported boundary conditions at $y=0$ and L, as an example, the global spectral matrix equation can be condensed in the form of

$$\begin{Bmatrix} \Delta\bar{w}_1' \\ \Delta\bar{w}_2 \\ \Delta\bar{w}_2' \\ \Delta\bar{w}_3' \end{Bmatrix} = \begin{bmatrix} \sigma_{22}^1 & \sigma_{23}^1 & \sigma_{24}^1 & 0 \\ \sigma_{32}^1 & \sigma_{33}^1+\sigma_{11}^2 & \sigma_{34}^1+\sigma_{12}^2 & \sigma_{14}^2 \\ \sigma_{42}^1 & \sigma_{43}^1+\sigma_{21}^2 & \sigma_{44}^1+\sigma_{22}^2 & \sigma_{24}^2 \\ 0 & \sigma_{41}^2 & \sigma_{42}^2 & \sigma_{44}^2 \end{bmatrix}^{-1} \begin{Bmatrix} 0 \\ \bar{f}(\xi ; k_x, \omega)\Delta\xi \\ 0 \\ 0 \end{Bmatrix} \quad (15)$$

or, simply

$$\{\Delta \bar{W}\}=[\Xi(k_x,\ \omega)]^{-1}\{\Delta \bar{F}\} \tag{16}$$

where $[\Xi]$ is the condensed global spectral matrix, $\{\Delta \bar{W}\}$ the global spectral nodal DOF vector, and $\{\Delta \bar{F}\}$ is the global spectral force vector. In Eq. (15), the superscripts (1 and 2) used for the spectral element components, σ, indicate the number of spectral elements and the subscripts (1 to 4) indicate the component number of the corresponding spectral element matrix. Also, the subscripts 1, 2, and 3 used in $\{\Delta \bar{W}\}$ indicate the node lines at $y=0$, $y=\xi$, and $y=L$, respectively, as shown in Fig. 2(c). Note that the spectral nodal DOF vector $\{\Delta \bar{W}\}$ is a function of the characteristic wave number k_x and frequency ω.

Following exactly the same procedure as used in the one-dimensional (beam) problem (Lee and Lee, 1996), the dynamic response at a point $(x,\ y)$ due to a line load acting at $y=\xi$ can be obtained from

$$\begin{aligned}\Delta \bar{w}(x,\ y\ ;\ \xi,\ k_x,\ \omega)=\bar{X}(x)[Q(y\ ;\ k_x,\ \omega)]\cdot\\ [a(k_x,\ \omega)]^{-1}[c][\Xi(k_x,\ \omega)]^{-1}\{d\}\cdot\\ \bar{f}(\xi\ ;\ k_x,\ \omega)\Delta\xi\end{aligned} \tag{17}$$

where $[Q]$, $[a]$, and $[\Xi]$ are defined in Eqs. (10), (11), and (16). The matrix $[c]$ relates the spectral nodal DOF $\{\Delta \bar{w}\}$ to $\{\Delta \bar{W}\}$. Similarly, the vector $\{d\}$ is determined from the force vector $\{\Delta \bar{F}\}$ itself, as explained in the beam problem (Lee and Lee, 1996).

Equation (17) gives the dynamic response due to an equivalent line load, thus the total dynamic response can be obtained from

$$\bar{w}(x,\ y\ ;\ k_x,\ \omega)=\int_{a=\xi/L}^{b=3\xi/L}\Delta \bar{w}(x,\ y\ ;\ \xi,\ k_x,\ \omega) \tag{18}$$

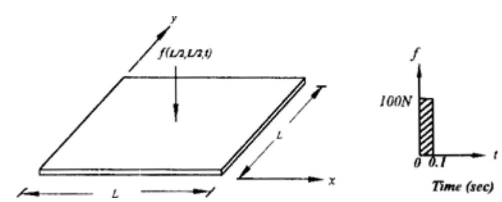
A Trapezoidal numerical integration algorithm (Chapra and Canle, 1989) may be applied to Eq. (18).

One notes that the dynamic response given by Eq. (18) is obtained in both the frequency ω and wave number k_x domains. Thus the dynamic response in the time domain can be obtained from the IFFT algorithm, first with respect to the wave number k_x, and then with respect to frequency ω as follows :

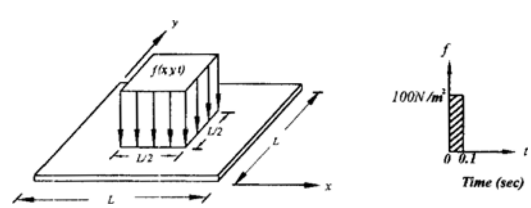
$$w(x,y,t)=IFFT_{\omega n}[IFFT_{kxm}\{\bar{w}_{mn}(x,y\ ;\ k_{xm},\ \omega_n)\}] \tag{19}$$

5. Illustrative Examples

Consider a simply-supported square plate having the dimension $L=2.8\ m$ and thickness $h=0.01\ m$ as shown in Fig. 3. Material properties of the plate are given as $D=6.73\ kN$, $m=28 kg/m^2$, and the loss factor is $\eta=0.03$. Two types of dynamic load are considered in this study : a centered point load of magnitude 100 N (see Fig. 3(a)), and a centered square load of magnitude of



(a) Point Load



(b) Distributed Load

Fig. 3 Examples of simply-supported square plates subject to (a) a centered point load of magnitude 100 N, and (b) a centered square dynamic load of magnitude 100 N/m^2 uniformly distributed over 1/4 area of the plate, both act for 0.1 seconds.

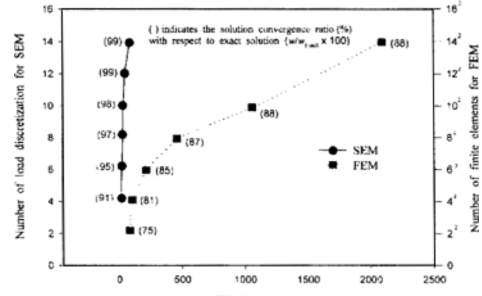


Fig. 4 Comparison of the solution convergence ratio and CPU times for SEM and FEM.

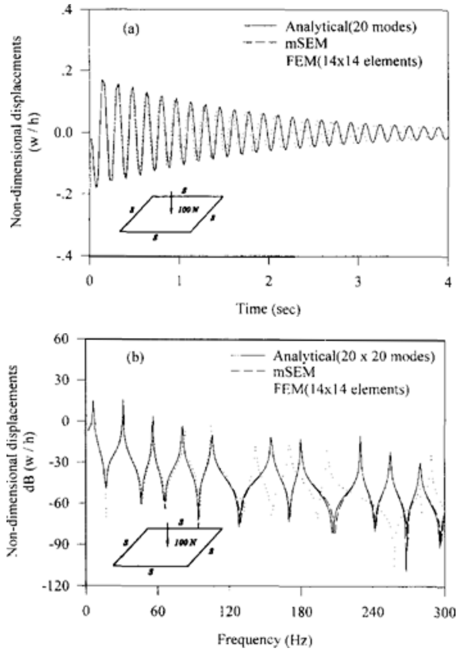


Fig. 5 Comparison of the dynamic and frequency responses at the middle of a square plate subject to a centered point load of magnitude 100 N, acting for 0.1 seconds : (a) dynamic responses, and (b) frequency responses.

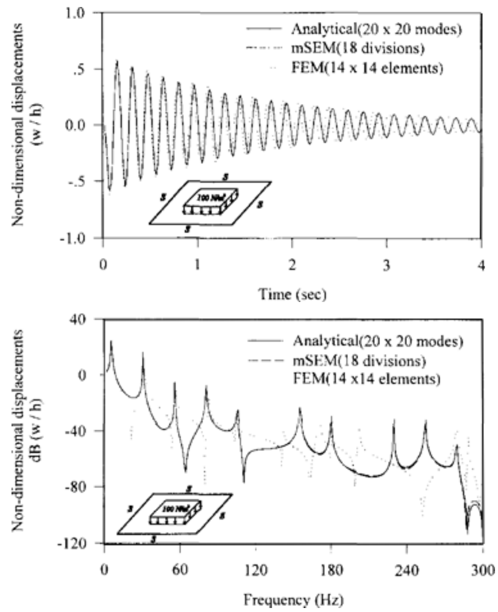


Fig. 6 Comparison of the dynamic and frequency responses at the middle of a square plate subject to a centered square dynamic load of magnitude 100 N/m^2, uniformly distributed over 1/4 area of the plate and acting for 0.1 seconds : (a) dynamic responses, and (b) frequency responses

100 N/m^2 uniformly distributed over a quarter area of the plate (see Fig. 3(b)). Both loads act for 0.1 seconds. The non-dimensional displacement responses (time histories and frequency response functions) by three different solution methods are compared in Figs. (5) and (6). Comparisons are made for the solutions which sufficiently converged within 0.03%, as shown in Fig. (4)

In general, the solutions by the present SEM are very close to the analytical exact solutions. The finite element solutions deviate significantly from the exact solutions with time, especially at high frequency. We have algo observed that substantial savings in computer time are achieved for obtaining solutions that converge to within 0.03% when the present SEM is used, in faver of FEM.

6. Conclusions

In this paper, a modified spectral element method is developed for rectangular plates subjected to distributed dynamic loads. Numerical tests show that the present SEM provides very accurate solutions when compared with finite element solutions. Since the present SEM is applicable only to rectangular plates, it is mandatory to develop a further generalized SEM for plates with arbitrary geometry and boundary conditions. This issue is a topic of ongoing research and the results will be discussed in a future paper.

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Appendix

Elements of matrix $[\alpha_{ij}]$:

$$\alpha_{11}=\alpha_{12}=\alpha_{13}=\alpha_{14}=1$$

$$\alpha_{21}=-\alpha_{22}=ik_y \qquad \alpha_{23}=-\alpha_{24}=k_y$$

$$\alpha_{31}=\alpha_{32}^{-1}=e^{ik_y l} \qquad \alpha_{33}=\alpha_{34}^{-1}=e^{k_y l}$$

$$\alpha_{41}=ik_y e^{ik_y l} \qquad \alpha_{42}=-ik_y e^{-ik_y l}$$

$$\alpha_{43}=k_y e^{k_y l} \qquad \alpha_{44}=-k_y e^{-k_y l}$$

Elements of matrix $[\beta_{ij}]$:

$$\beta_{11}=-\beta_{12}=-iDk_y\{k_y^2+(2-\nu)k_x^2\}$$

$$\beta_{13}=-\beta_{14}=Dk_y\{k_y^2-(2-\nu)k_x^2\}$$

$$\beta_{21}=\beta_{22}=-D\{k_y^2+\nu k_x^2\}$$

$$\beta_{23}=\beta_{24}=D\{k_y^2-\nu k_x^2\}$$

$$\beta_{31}=-iDk_y\{k_y^2+(2-\nu)k_x^2\}e^{ik_y l}$$

$$\beta_{32}=iDk_y\{k_y^2+(2-\nu)k_x^2\}e^{-ik_y l}$$

$$\beta_{33}=Dk_y\{k_y^2-(2-\nu)k_x^2\}e^{k_y l}$$

$$\beta_{34}=-Dk_y\{k_y^2-(2-\nu)k_x^2\}e^{-k_y l}$$

$$\beta_{41}=-D\{k_y^2+\nu k_x^2\}e^{ik_y l}$$

$$\beta_{42}=-D\{k_y^2+\nu k_x^2\}e^{-ik_y l}$$

$$\beta_{43}=D\{k_y^2-\nu k_x^2\}e^{k_y l}$$

$$\beta_{44}=D\{k_y^2-\nu k_x^2\}e^{-k_y l}$$